

ON SOME TRIGONOMETRIC SUMMABILITY METHODS AND GIBBS' PHENOMENON

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1. **Introduction.** We consider Fourier sine series

$$(1.1) \quad f(\theta) \sim \sum_1^\infty b_\nu \sin \nu\theta, \quad \sum_1^n b_\nu \sin \nu\theta = s_n(\theta).$$

Suppose that $f(+0)$ exists and is greater than 0, that is $f(\theta)$ has a simple jump $2f(+0)$ at $\theta=0$. We say that the series (1.1) presents Gibbs' phenomenon, if

$$(1.2) \quad \limsup s_n(\theta_n) > f(+0), \text{ as } \theta_n \downarrow 0.$$

More generally, if only for some $k \geq 0$

$$\lim_{\theta \downarrow 0} \frac{2(k+1)}{\theta^{k+1}} \int_0^\theta (\theta-t)^k f(t) dt = j > 0$$

exists, and $\limsup s_n(\theta_n) > j/2$, we say that the series (1.1) presents a generalized Gibbs' phenomenon at $\theta=0$; j is the generalized jump of $f(\theta)$ at $\theta=0$. Our aim is to find general conditions for $f(\theta)$ or its Fourier coefficients, which imply a Gibbs' phenomenon. It is known that the jump j is closely connected with the asymptotic behavior of the sequence $\{nb_n\}$; on the other hand

$$s_n(\theta_n) = \sum_1^n \nu b_\nu \frac{\sin \nu\theta_n}{\nu}$$

is a linear transform of $\{nb_n\}$, with the triangular matrix $a_{\nu n} = \nu^{-1} \sin \nu\theta_n$, $\nu=1, 2, \dots, n$. In a previous paper [3]⁽¹⁾ we have discussed the relationship of the transform

$$T_n(\theta_n) = \sum_1^n \tau_\nu \frac{\sin \nu\theta_n}{\nu}$$

to the Cesàro means of the sequence $\{\tau_n\}$. As an application we have proved the presence of a Gibbs' phenomenon under general assumptions and we gave new formulae for the determination of the jump $2f(+0)$. More general results are given in the present paper; the knowledge of (3) is not assumed.

We consider trigonometric linear transforms, related to T_n . We discuss in particular the transforms

Presented to the Society, April 24, 1943; received by the editors April 24, 1943.

⁽¹⁾ Numbers in brackets refer to the literature cited at the end of this paper.

$$B_n = \frac{1}{2} \{ T_n(\theta_n) + T_n(\phi_n) \}, \quad H_n = \frac{1}{n} \sum_{k=1}^n T_k(\theta_k),$$

$$S_n = \frac{1}{\theta_n} \int_0^{\theta_n} T_n(t) dt.$$

In §5 we prove a theorem on $(C, 2)$ summability of the sequence $\{nb_n\}$ for Fourier series.

We shall make repeated use of the well known theorem:

THEOREM A. *The convergence of a sequence $\{\tau_n\}$ implies the convergence of the transform $T_n = \sum_1^n a_{n\nu} \tau_\nu$, if and only if*

$$\lim_{n \rightarrow \infty} a_{n\nu} = a_\nu \text{ exists for } \nu = 1, 2, 3, \dots,$$

$$\sum_{\nu=1}^n |a_{n\nu}| = O(1), \text{ as } n \rightarrow \infty,$$

$$\lim_{n \rightarrow \infty} \sum_{\nu=1}^n a_{n\nu} = \sigma \text{ exists.}$$

We then have $\lim T_n = \sigma \lim \tau_n + \sum_1^\infty a_\nu (\tau_\nu - \lim \tau_n)$ ⁽²⁾.

2. The average $(1/2)\{T_n(\theta_n) + T_n(\phi_n)\} = B_n$. On putting $\rho_n = 1$ in Theorem 2 of our paper [3] we get easily the following theorem.

THEOREM 1. *Suppose that*

$$(2.1) \quad \frac{1}{n} \sum_1^n \tau_\nu \rightarrow \tau, \quad \phi_n > 0, \quad \theta_n \geq 0, \quad n\theta_n = O(1), \quad n\phi_n = O(1),$$

and that $\sigma(\alpha) = \int_0^\alpha (t^{-1} \sin t) dt$ has the same value for all limit points α of the sequence $\{n\theta_n\}$, and the same value for all limit points β of the sequence $\{n\phi_n\}$; then

$$(2.2) \quad \lim_{n \rightarrow \infty} B_n = (\tau/2)(\sigma(\alpha) + \sigma(\beta)).$$

For the special case $\phi_n = \pi/n$ see Rogosinski [1, 2].

We next assume $(C, 2) \lim \tau_n = \tau$. Let $\tau'_n = \sum_1^n \tau_\nu$, $\tau_n^2 = \sum_1^n \tau_\nu^2$, then our assumption is: $2\tau_n^2/n^2 \rightarrow \tau$. We write $\Delta^0 \tau_n = \tau_n$, $\Delta \tau_n = \tau_n - \tau_{n+1}$, $\Delta^2 \tau_n = \Delta(\Delta \tau_n)$; then

$$(2.3) \quad B_n = \sum_1^{n-2} \tau_\nu \Delta^2 \frac{\sin \nu \theta_n + \sin \nu \phi_n}{\nu} + \tau_{n-1}^2 \left\{ \frac{\sin (n-1)\theta_n + \sin (n-1)\phi_n}{n-1} \right.$$

$$\left. - 2 \frac{\sin n\theta_n + \sin n\phi_n}{n} \right\} + \tau_n^2 \frac{\sin n\theta_n + \sin n\phi_n}{n}.$$

⁽²⁾ The statement at the beginning of §2 in [3] should be corrected accordingly; it does not affect the rest of the paper.

It follows from Theorem A that, if in addition to the assumptions of Theorem 1,

$$\sum_1^{n-2} \nu^2 \left| \Delta^2 \frac{\sin \nu \theta_n + \sin \nu \phi_n}{\nu} \right| + n^2 \left| \Delta \frac{\sin (n-1)\theta_n + \sin (n-1)\phi_n}{n-1} \right| + n |\sin n\theta_n + \sin n\phi_n| = O(1),$$

then again (2.2) holds.

We now assume $n(\phi_n - \theta_n) = (2\mu - 1)\pi + \delta_n$, $\delta_n = O(1/n)$, μ an integer ≥ 1 . Then

$$\sin n\theta_n + \sin n\phi_n = 2 \sin \frac{n}{2} (\theta_n + \phi_n) \cos \frac{n}{2} (\theta_n - \phi_n) = O\left(\frac{1}{n}\right);$$

also

$$\begin{aligned} \cos \frac{n-1}{2} (\theta_n - \phi_n) &= \cos \frac{n}{2} (\theta_n - \phi_n) \cos \frac{\theta_n - \phi_n}{2} \\ &\quad + \sin \frac{n}{2} (\theta_n - \phi_n) \sin \frac{\theta_n - \phi_n}{2} = O\left(\frac{1}{n}\right). \end{aligned}$$

Furthermore, for $\theta > 0$,

$$\Delta^2 \left(\frac{\sin \nu \theta}{\nu} \right) = \Delta^2 \int_0^\theta \cos \nu t dt = R \int_0^\theta \Delta^2 z^r dt, \quad z = c^{it}.$$

Hence

$$\Delta^2 \frac{\sin \nu \theta}{\nu} = R \int_0^\theta z^r (1-z)^2 dt,$$

and

$$\begin{aligned} (2.4) \quad \left| \Delta^2 \frac{\sin \nu \theta}{\nu} \right| &< \int_0^\theta |1-z|^2 dt = \int_0^\theta |e^{-it/2} - e^{it/2}|^2 dt \\ &= 4 \int_0^\theta \sin^2 \frac{t}{2} dt < \int_0^\theta t^2 dt = \frac{\theta^3}{3}. \end{aligned}$$

Thus in view of (2.1)

$$\sum_1^{n-2} \nu^2 \left| \Delta^2 \frac{\sin \nu \theta_n + \sin \nu \phi_n}{\nu} \right| < (\theta_n^3 + \phi_n^3) \sum_1^n \nu^2 = O(1).$$

We have thus proved the following theorem.

THEOREM 2. Suppose that $(C, 2) \lim \tau_n = \tau$, $\phi_n > 0$, $\theta_n \geq 0$, $n\theta_n = O(1)$, $n\phi_n = n\theta_n + (2\mu - 1)\pi + O(1/n)$, μ an integer greater than or equal to 1, and that $\sigma(\alpha) = \int_0^\alpha t^{-1} \sin t dt$ has the same value for all limit points α of the sequence $\{n\theta_n\}$. Then $\lim B_n = (\tau/2) \{ \sigma(\alpha) + \sigma(\alpha + (2\mu - 1)\pi) \}$.

3. The average $(1/n) \sum_{k=1}^n T_k(\theta_k) = H_n$. We have

$$(3.1) \quad nH_n = \sum_{\kappa=1}^n \sum_{\nu=1}^{\kappa} \nu^{-1} \tau_{\nu} \sin \nu \theta_{\kappa} = \sum_{\nu=1}^n \nu^{-1} \tau_{\nu} \sum_{\kappa=\nu}^n \sin \nu \theta_{\kappa} = \sum_{\nu=1}^n C_{n\nu} \tau_{\nu},$$

where

$$(3.2) \quad C_{n\nu} = \nu^{-1} \sum_{\kappa=\nu}^n \sin \nu \theta_{\kappa}, \quad \nu = 1, 2, \dots, n.$$

Thus $nH_n = \sum_{\nu=1}^{n-1} (C_{n\nu} - C_{n,\nu+1}) \tau_{\nu} + C_{nn} \tau'_n$. We first prove the following theorem.

THEOREM 3. If $(C, 1) \lim \tau_n = \tau$, and

$$(3.3) \quad \frac{1}{n} \sum_1^n |\nu \theta_{\nu} - \alpha| \rightarrow 0,$$

then

$$(3.4) \quad H_n \rightarrow \tau \int_0^{\alpha} \frac{\sin t}{t} dt.$$

(3.3) is strong summability $(C, 1)$ of the sequence $\{n\theta_n\}$ to α ; it implies that $\theta_n \rightarrow 0$, hence $\lim_{n \rightarrow \infty} n^{-1} C_{n\nu} = 0$ for each ν . Furthermore

$$\begin{aligned} & \sum_1^{n-1} \nu |C_{n\nu} - C_{n,\nu+1}| + n |C_{nn}| \\ &= \sum_1^{n-1} \left| \sum_{\kappa=\nu}^n \sin \nu \theta_{\kappa} - \frac{\nu}{\nu+1} \sum_{\nu+1}^n \sin (\nu+1) \theta_{\kappa} \right| + |\sin n \theta_n| \\ &< \sum_1^{n-1} |\sin \nu \theta_{\nu}| + \sum_1^{n-1} \left| \sum_{\kappa=\nu+1}^n (\sin \nu \theta_{\kappa} - \sin (\nu+1) \theta_{\kappa}) \right| \\ &\quad + \sum_1^{n-1} \frac{1}{\nu+1} \left| \sum_{\nu+1}^n \sin (\nu+1) \theta_{\kappa} \right| + 1 \\ &< \sum_1^n \nu \theta_{\nu} + 2 \sum_1^{n-1} \sum_{\nu+1}^n \left| \sin \frac{\theta_{\kappa}}{2} \cos \frac{2\nu+1}{2} \theta_{\kappa} \right| + \sum_1^{n-1} \sum_{\nu+1}^n \theta_{\kappa} + 1 \\ &< 3 \sum_1^n \nu \theta_{\nu} + 1 = O(n), \quad \text{as } n \rightarrow \infty, \end{aligned}$$

hence $H_n = O(1)$. Thus, in order that H_n has a limit whenever $n^{-1} \tau'_n$ tends to a limit τ , the additional condition is that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\nu=1}^n C_{n\nu} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\nu=1}^n \nu^{-1} \sum_{\kappa=\nu}^n \sin \nu \theta_{\kappa} = \sigma$$

exists. We then have $H_n \rightarrow \sigma \tau$ (cf. Theorem A). Now

$$\frac{1}{n} \sum_1^n C_{n\nu} = \frac{1}{n} \sum_{\kappa=1}^n S_{\kappa}(\theta_{\kappa}),$$

and

$$\begin{aligned} S_{\kappa}(\theta_{\kappa}) &= \sum_{\lambda=1}^{\kappa} \lambda^{-1} \sin \lambda \theta_{\kappa} = \int_0^{\theta_{\kappa}} \left(\sum_1^{\kappa} \cos \lambda t \right) dt \\ &= -\frac{\theta_{\kappa}}{2} + \int_0^{\theta_{\kappa}} \frac{\sin (\kappa + 1/2)t}{2 \sin (t/2)} dt. \end{aligned}$$

Furthermore

$$\begin{aligned} \int_0^{\theta} \frac{\sin (\kappa + 1/2)t}{2 \sin (t/2)} dt &= \int_0^{\theta} \frac{\sin (\kappa + 1/2)t}{t} dt \\ &\quad + \int_0^{\theta} \sin (\kappa + 1/2)t \left\{ \frac{1}{2 \sin (t/2)} - \frac{1}{t} \right\} dt, \end{aligned}$$

and, from the mean value theorem,

$$(3.5) \quad 0 < \frac{1}{2 \sin (t/2)} - \frac{1}{t} = \frac{t - 2 \sin (t/2)}{2t \sin (t/2)} < \frac{t}{12}, \quad \theta < t < \pi.$$

Hence

$$\left| \int_0^{\theta} \sin (\kappa + 1/2)t \left\{ \frac{1}{2 \sin (t/2)} - \frac{1}{t} \right\} dt \right| < \frac{1}{12} \int_0^{\theta} t dt = \frac{1}{24} \theta^2,$$

and

$$\begin{aligned} S_{\kappa}(\theta_{\kappa}) &= -\frac{\theta_{\kappa}}{2} + \int_0^{(\kappa+1/2)\theta_{\kappa}} \frac{\sin t}{t} dt + \delta_{\kappa} \theta_{\kappa}^2 \\ &= \int_0^{\alpha} \frac{\sin t}{t} dt + \int_{\alpha}^{(\kappa+1/2)\theta_{\kappa}} \frac{\sin t}{t} dt - \frac{\theta_{\kappa}}{2} + \delta_{\kappa} \theta_{\kappa}^2, \end{aligned}$$

where $|\delta_{\kappa}| < 1/24$. Thus, for $0 < \theta_{\kappa} < 1$,

$$\left| S_{\kappa}(\theta_{\kappa}) - \int_0^{\alpha} \frac{\sin t}{t} dt \right| < |(\kappa + 1/2)\theta_{\kappa} - \alpha| + \theta_{\kappa} < |\kappa \theta_{\kappa} - \alpha| + 2\theta_{\kappa},$$

and from (3.3)

$$\left| \frac{1}{n} \sum_1^n S_{\kappa}(\theta_{\kappa}) - \int_0^{\alpha} \frac{\sin t}{t} dt \right| < \frac{1}{n} \sum_1^n |\kappa \theta_{\kappa} - \alpha| + \frac{2}{n} \sum_1^n \theta_{\kappa} \rightarrow 0;$$

which proves Theorem 3. For the case of $(C, 2)$ summability we prove the following theorem.

THEOREM 4. *If $(C, 2) \lim \tau_n = \tau$,*

$$(3.6) \quad \sum_1^n \nu^2 |\theta_\nu - \theta_{\nu+1}| = O(n), \quad \text{as } n \rightarrow \infty,$$

and if (3.3) holds, then (3.4) holds.

Evidently (3.3) implies $\sum_1^n \nu \theta_\nu = O(n)$; furthermore

$$(3.7) \quad \sum_1^n \nu^2 (\theta_\nu - \theta_{\nu+1}) = \sum_1^n (2\nu - 1)\theta_\nu - n^2 \theta_{n+1} = O(n) - n^2 \theta_{n+1},$$

hence from (3.6)

$$(3.8) \quad n\theta_n = O(1) \quad \text{as } n \rightarrow \infty.$$

We have from (3.1)

$$nH_n = \sum_1^{n-2} \tau_\nu'' \Delta^2 C_{n\nu} + \tau_{n-1}'' (C_{n,n-1} - 2C_{nn}) + \tau_n'' C_{nn},$$

where $nC_{nn} = \sin n\theta_n = O(1)$, and

$$n(C_{n,n-1} - 2C_{nn}) = O(1) + (n/(n-1)) \{ \sin(n-1)\theta_{n-1} + \sin(n-1)\theta_n \} = O(1).$$

Furthermore, using (2.4) and (3.8),

$$\begin{aligned} & \sum_1^{n-2} \nu^2 |\Delta^2 C_{n\nu}| \\ &= \sum_1^{n-2} \nu^2 \left| \frac{1}{\nu} \sum_{\kappa=\nu}^n \sin \nu \theta_\kappa - \frac{2}{\nu+1} \sum_{\kappa=\nu+1}^n \sin(\nu+1)\theta_\kappa + \frac{1}{\nu+2} \sum_{\kappa=\nu+2}^n \sin(\nu+2)\theta_\kappa \right| \\ &\leq \sum_1^{n-2} \nu^2 \left| \frac{1}{\nu} (\sin \nu \theta_\nu + \sin \nu \theta_{\nu+1}) - \frac{2}{\nu+1} \sin(\nu+1)\theta_{\nu+1} \right| \\ &\quad + \sum_1^{n-2} \nu^2 \left| \sum_{\kappa=\nu+2}^n \left(\frac{\sin \nu \theta_\kappa}{\nu} - \frac{2 \sin(\nu+1)\theta_\kappa}{\nu+1} + \frac{\sin(\nu+2)\theta_\kappa}{\nu+2} \right) \right| \\ &< O(n) + \sum_1^{n-1} \nu |\sin \nu \theta_\nu + \sin \nu \theta_{\nu+1} - 2 \sin(\nu+1)\theta_{\nu+1}| \\ &\quad + \sum_1^{n-2} \nu^2 \sum_{\kappa=\nu+2}^n |\Delta^2 \nu^{-1} \sin \nu \theta_\kappa| \\ &< O(n) + \sum_1^{n-1} \nu (\theta_{\nu+1} + \theta_{\nu+1} + \nu |\theta_\nu - \theta_{\nu+1}|) + \sum_1^n \nu^2 \left(\sum_{\kappa=\nu}^n \theta_\kappa^3 \right) \\ &= O(n) + O\left(\sum_1^n \nu^3 \theta_\nu^3 \right) = O(n). \end{aligned}$$

Now (3.4) follows as in Theorem 3. This proves Theorem 4. In this connection the following lemma is of interest:

LEMMA. *If for some $c > 0$*

$$(3.9) \quad 0 \leq \theta_{n+1} \leq (1 + c/n)\theta_n, \quad n = 1, 2, 3, \dots,$$

and

$$(3.10) \quad \sum_1^n \nu \theta_\nu = O(n), \quad \text{as } n \rightarrow \infty,$$

then (3.6) and (3.8) hold.

We write $\sum_1^n \nu^2 |\theta_\nu - \theta_{\nu+1}| = \sum' + \sum''$, where $\sum'$ contains all terms with $\theta_\nu \leq \theta_{\nu+1}$, and $\sum''$ the rest. Now, using (3.9) and (3.10),

$$\sum' \leq c \sum_1^n \nu \theta_\nu = O(n),$$

and from (3.7)

$$\sum'' - \sum' < \sum_1^n (2\nu - 1)\theta_\nu = O(n),$$

hence $\sum'' = O(n)$ and (3.6) is proved. This and (3.10) yield (3.8). In particular in Theorem 4 assumption (3.6) can be replaced by $\theta_n \downarrow$.

4. **The integral mean** $(1/\theta_n) \int_0^{\theta_n} (\sum_1^n \tau_\nu \nu^{-1} \sin \nu t) dt$. To a given sequence $\{\tau_n\}$ we consider the transform

$$(4.1) \quad S_n = \frac{1}{\theta_n} \int_0^{\theta_n} \left(\sum_1^n \tau_\nu \sin \nu t \right) dt = \sum_1^n \frac{1 - \cos \nu \theta_n}{\nu^2 \theta_n} \tau_\nu = \sum_1^n a_{n\nu} \tau_\nu, \quad \theta_n \rightarrow 0;$$

so that now

$$(4.2) \quad a_{n\nu} = (1 - \cos \nu \theta_n) / \nu^2 \theta_n, \quad \nu = 1, 2, \dots, n.$$

Evidently $0 < a_{n\nu} \rightarrow 0$ as $n \rightarrow \infty$; hence a necessary and sufficient condition that for every convergent sequence $\tau_n \rightarrow \tau$ the sequence S_n has a limit $\sigma\tau$ is that $\lim \sum_1^n A_{n\nu} = \sigma$ exists (from Theorem A). Now

$$\begin{aligned} \sum_1^n \frac{1 - \cos \nu \theta_n}{\nu^2} &= \int_0^{\theta_n} \left(\sum_1^n \frac{\sin \nu t}{\nu} \right) dt \\ &= \int_0^{\theta_n} \left\{ -\frac{t}{2} + \int_0^t \frac{\sin (n + 1/2)u}{2 \sin (u/2)} du \right\} dt \\ &= -\frac{\theta_n^2}{4} + \int_0^{\theta_n} (\theta_n - t) \frac{\sin (n + 1/2)t}{2 \sin (t/2)} dt, \end{aligned}$$

thus

$$\begin{aligned} \sum_1^n a_{nv} &= -\frac{\theta_n^2}{4} + \theta_n^{-1} \int_0^{\theta_n} (\theta_n - t) \frac{\sin (n + 1/2)t}{t} dt \\ &\quad + \theta_n^{-1} \int_0^{\theta_n} (\theta_n - t) \left\{ \frac{1}{2 \sin (t/2)} - \frac{1}{t} \right\} \sin (n + 1/2)t dt, \end{aligned}$$

and using (3.5)

$$\begin{aligned} \theta_n^{-1} \left| \int_0^{\theta_n} (\theta_n - t) \left\{ \frac{1}{2 \sin (t/2)} - \frac{1}{t} \right\} \sin (2n + 1) \frac{t}{2} dt \right| \\ < \theta_n^{-1} \int_0^{\theta_n} (\theta_n - t) \frac{tdt}{12} < \frac{1}{24} \theta_n^2 \rightarrow 0. \end{aligned}$$

Hence

$$\begin{aligned} \sum_1^n a_{nv} &= o(1) + \theta_n^{-1} \int_0^{(n+1/2)\theta_n} \left(\theta_n - \frac{t}{n+1/2} \right) \frac{\sin t}{t} dt \\ &= o(1) + \int_0^{(n+1/2)\theta_n} \frac{\sin t}{t} dt - \frac{2}{(2n+1)\theta_n} [1 - \cos (n+1/2)\theta_n]. \end{aligned}$$

If $\beta \leq \infty$ is a limit point of the sequence $\{n\theta_n\}$, then for a subsequence of integers n

$$\sum_1^n a_{nv} \rightarrow \int_0^\beta \frac{\sin t}{t} dt - \frac{1 - \cos \beta}{\beta} = g(\beta).$$

Now $g'(\beta) = (1 - \cos \beta)/\beta^2 \geq 0$, hence $g(\beta) \uparrow$, as β increases, to $g(\infty) = \pi/2$. We have thus proved:

THEOREM 5. *In order that for every convergent sequence $\tau_n \rightarrow \tau$ the transform $S_n = \sum_1^n \tau_v (1 - \cos v\theta_n)/v^2\theta_n$ has a limit, it is necessary and sufficient that $\lim_{n \rightarrow \infty} \theta_n = \beta$, $\beta \leq \infty$, exists. We then have $S_n \rightarrow \tau g(\beta)$, where $g(\beta) = \int_0^\beta (t^{-1} \sin t) dt - (1 - \cos \beta)/\beta \leq \pi/2$.*

We now assume only $n^{-1} \sum_1^n \tau_v \rightarrow \tau$, or $n^{-1} \tau'_n \rightarrow \tau$. We have $S_n = \sum_1^{n-1} (a_{nv} - a_{n,v+1}) \tau'_v + a_{nn} \tau'_n$; and the additional condition for the existence of $\lim S_n$ is by Theorem A:

$$\sum_1^{n-1} v \left| \frac{1 - \cos v\theta_n}{v^2\theta_n} - \frac{1 - \cos (v+1)\theta_n}{(v+1)^2\theta_n} \right| + \frac{1 - \cos n\theta_n}{n\theta_n} = O(1),$$

or

$$(4.3) \quad \theta_n^{-1} \sum_1^{n-1} v^{-3} | (v+1)^2 (1 - \cos v\theta_n) - v^2 (1 - \cos (v+1)\theta_n) | = O(1).$$

But

$$\theta_n^{-1} \sum_1^n \nu^{-3} (2\nu + 1) (1 - \cos \nu \theta_n) < 3 \sum_1^n \frac{1 - \cos \nu \theta_n}{\nu^2 \theta_n} = 3 \sum_1^n a_{n\nu} = O(1),$$

hence (4.3) reduces to $\theta_n^{-1} \sum_1^{n-1} \nu^{-1} |\cos \nu \theta_n - \cos (\nu+1)\theta_n| = O(1)$, as $n \rightarrow \infty$, or $\sum_1^n \nu^{-1} |\sin (\nu+1/2)\theta_n| = O(1)$.

But this condition is equivalent to $n\theta_n = O(1)$ (cf. [3, §2]). Thus we have the following:

THEOREM 6. *The transform $S_n = \sum_1^n \tau_\nu (1 - \cos \nu \theta_n) / \nu^2 \theta_n$ has a limit for every sequence $\{\tau_n\}$ which is summable $(C, 1)$ to τ , if and only if $\lim n\theta_n = \beta < \infty$ exists. We then have $S_n \rightarrow \tau g(\beta)$.*

We finally assume $(C, 2) \lim \tau_n = \tau$; using the formula

$$\sum_1^n a_\nu \tau_\nu = \sum_1^{n-2} \tau_\nu \Delta^2 a_\nu + \tau_{n-1} (a_{n-1} - 2a_n) + \tau_n a_n,$$

we now get for the existence of $\lim S_n$ the additional conditions

$$(4.4) \quad \theta_n^{-1} \sum_1^{n-2} \nu^2 \left| \frac{1 - \cos \nu \theta_n}{\nu^2} - 2 \frac{1 - \cos (\nu+1)\theta_n}{(\nu+1)^2} + \frac{1 - \cos (\nu+2)\theta_n}{(\nu+2)^2} \right| = O(1),$$

$$(4.5) \quad \theta_n^{-1} n^2 \left| \frac{1 - \cos (n-1)\theta_n}{(n-1)^2} - 2 \frac{1 - \cos n\theta_n}{n^2} \right| + \frac{1 - \cos n\theta_n}{\theta_n} = O(1).$$

In particular $\sin^2 n\theta_n/2 = O(\theta_n) = o(1)$, hence $n\theta_n \rightarrow \beta = 2\lambda\pi$, λ an integer. We assume $\lambda > 0$; on putting $n\theta_n = 2\lambda\pi + 2\epsilon_n$, $\epsilon_n \rightarrow 0$, we must have

$$\sin^2 (\lambda\pi + \epsilon_n) = \sin^2 \epsilon_n = O((\lambda\pi + \epsilon_n)/n) = O(1/n),$$

or $\epsilon_n = O(1/n^{1/2})$. To satisfy (4.5) the additional condition is $\sin^2 (n-1)\theta_n = O(\theta_n)$, or $(\sin n\theta_n \cos \theta_n - \cos n\theta_n \sin \theta_n)^2 = O(1/n)$, which reduces to our previous condition. We finally show that now (4.4) is also satisfied. We have

$$\frac{1 - \cos \nu \theta}{\nu^2} = \int_0^\theta dt \int_0^t \cos \nu u du = \int_0^\theta (\theta - t) \cos \nu t dt,$$

hence

$$\Delta^2 \frac{1 - \cos \nu \theta}{\nu^2} = \int_0^\theta (\theta - t) \Delta^2 \cos \nu t dt = R \int_0^\theta (\theta - t) z^\nu (1 - z)^2 dt, \quad z = e^{it},$$

and

$$\left| \Delta^2 \frac{1 - \cos \nu \theta}{\nu^2} \right| < \int_0^\theta (\theta - t) |1 - e^{it}|^2 dt < \int_0^\theta (\theta - t) t^2 dt = \frac{\theta^4}{12}.$$

Using this inequality, the left side of (4.4) is less than $\sum_1^{n-2} \nu^2 \theta_n^3 < n^3 \theta_n^3 = O(1)$. This proves the following theorem.

THEOREM 7. *The transform S_n has a limit $\sigma\tau$ (with $\sigma \neq 0$) for every sequence $\{\tau_n\}$ which is summable $(C, 2)$ to τ if and only if $n\theta_n = 2(\lambda\pi + \epsilon_n)$, $\epsilon_n = O(n^{-1/2})$, λ an integer greater than 0. We then have $S_n \rightarrow \tau \int_0^{2\lambda\pi} t^{-1} \sin t \, dt = \tau\sigma(2\lambda\pi)$.*

5. $(C, 2)$ summability of $\{nb_n\}$ for sine series. We shall prove the following theorem.

THEOREM 8. *If $f(\theta) \sim \sum_1^\infty b_\nu \sin \nu\theta$, and for some j*

$$(5.1) \quad \int_0^\theta |f(t) - j/2| \, dt = O(\theta), \quad \int_0^\theta \{f(t) - j/2\} \, dt = o(\theta), \quad \text{as } \theta \downarrow 0,$$

then

$$(5.2) \quad (C, 2) \lim nb_n = j/\pi.$$

We have, for $\tau_n = nb_n$,

$$\tau_n^2 = \sum_1^n (n - \nu + 1) \nu b_\nu = \frac{2}{\pi} \int_0^\pi f(t) \left(\sum_1^n (n - \nu + 1) \nu \sin \nu t \right) dt,$$

and $\sum_1^n (n - \nu + 1) \nu \sin \nu t = -(d/dt) \sum_1^n (n - \nu + 1) \cos \nu t$. On putting

$$\sum_1^n \cos \nu t = -\frac{1}{2} + \frac{\sin(n + 1/2)t}{2 \sin(t/2)} \equiv \sigma_n(t),$$

we have

$$\begin{aligned} \sum_1^n (n - \nu + 1) \cos \nu t &= \sum_1^n \sigma_\nu(t) = -\frac{n}{2} + \frac{1}{2 \sin(t/2)} \sum_1^n \sin(\nu + 1/2)t \\ &= -\frac{n}{2} + \frac{1}{2 \sin(t/2)} \{ \sin^2(n + 1)(t/2) - \sin^2(t/2) \} \\ &= -\frac{n + 1}{2} + \frac{1}{2} \cdot \frac{\sin^2(n + 1)(t/2)}{\sin^2(t/2)}, \end{aligned}$$

hence

$$\begin{aligned} (5.3) \quad & \sum_1^n (n - \nu + 1) \nu \sin \nu t \\ &= \frac{1}{2} \frac{\sin(n + 1)(t/2)}{\sin^3(t/2)} \left\{ \cos \frac{t}{2} \sin(n + 1) \frac{t}{2} - (n + 1) \cos(n + 1) \frac{t}{2} \sin \frac{t}{2} \right\}. \end{aligned}$$

Thus

$$(5.4) \quad \tau_n^2 = \frac{1}{\pi} \int_0^\pi f(t) \frac{\sin (n+1)(t/2)}{\sin^3 (t/2)} \left\{ \sin n \frac{t}{2} - n \cos (n+1) \frac{t}{2} \sin \frac{t}{2} \right\} dt;$$

furthermore

$$(5.5) \quad \begin{aligned} & \int_0^\pi \left(\sum_1^n (n - \nu + 1) \nu \sin \nu t \right) dt \\ &= \sum_1^n (n - \nu + 1) \{1 - (-1)^\nu\} = 2 \sum_{\lambda=0}^{(n-1)/2} (n - 2\lambda) \\ &= 4 \sum_{\lambda=0}^{(n-1)/2} \left(\frac{n}{2} - \lambda \right) = h_n, \end{aligned}$$

say, where $2hn/n^2 \rightarrow 1$ as $n \rightarrow \infty$. Our aim is to prove

$$(5.6) \quad \tau_n^2/h_n \rightarrow j/\pi, \quad \text{or} \quad (\tau_n^2 - jh_n/\pi)/h_n \rightarrow 0.$$

From (5.3), (5.4) and (5.5)

$$\tau_n^2 - \frac{1}{\pi} j h_n = \frac{1}{\pi} \int_0^\pi \{f(t) - j/2\} \frac{\sin (n+1)(t/2)}{\sin^3 (t/2)} K_n(t) dt,$$

where

$$(5.7) \quad \begin{aligned} K_n(t) &= \sin (nt/2) - n \cos ((n+1)t/2) \sin (t/2) \\ &= n \sin (t/2) (1 - \cos ((n+1)t/2)) - (n \sin (t/2) - \sin (nt/2)). \end{aligned}$$

Now from (3.5)

$$\begin{aligned} 0 &< \frac{1}{\sin^3 (t/2)} - \frac{1}{(t/2)^3} = \left(\frac{1}{\sin (t/2)} - \frac{2}{t} \right) \left(\frac{1}{\sin^2 (t/2)} + \frac{2}{t \sin (t/2)} + \frac{4}{t^2} \right) \\ &< \frac{t}{6} \cdot \frac{3}{\sin^2 (t/2)} < \frac{\pi}{2 \sin (t/2)}, \end{aligned}$$

and

$$|K_n(t)| < n \sin (t/2) + |\sin nt/2|.$$

Hence

$$\begin{aligned} & \left| \int_0^\pi \{f(t) - j/2\} \left(\frac{1}{\sin^3 (t/2)} - \frac{1}{(t/2)^3} \right) \sin (n+1) \frac{t}{2} K_n(t) dt \right| \\ &< \frac{\pi}{2} \int_0^\pi |f(t) - j/2| \left(n + \frac{|\sin (nt/2)|}{\sin (t/2)} \right) dt = O(n) = o(h_n); \end{aligned}$$

thus (5.6) will follow if we prove

$$\int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2)}{t^3} K_n(t) dt = o(n^2).$$

In view of (5.7) this will follow from

$$(5.8) \quad \int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2)}{t^3} \sin \frac{t}{2} \left(1 - \cos(n+1) \frac{t}{2}\right) dt = o(n),$$

and

$$(5.9) \quad \int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2)}{t^3} \left(n \sin \frac{t}{2} - \sin n \frac{t}{2}\right) dt = o(n^2).$$

Again, using (3.5),

$$\begin{aligned} \left| \int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2)}{t^3} \left(\frac{t}{2} - \sin \frac{t}{2}\right) \left(1 - \cos(n+1) \frac{t}{2}\right) dt \right| \\ < \int_0^\pi |f(t) - j/2| dt = O(1), \end{aligned}$$

thus (5.8) is equivalent to

$$\int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2)}{t^2} \left(1 - \cos(n+1) \frac{t}{2}\right) dt = o(n),$$

or

$$I_n \equiv \int_0^\pi \{f(t) - j/2\} \frac{\sin((n+1)t/2) - 2^{-1} \sin(n+1)t}{t^2} dt = o(n).$$

We write $I_n = \int_0^{c/n} + \int_{c/n}^\pi = L_1 + L_2$, say, where c is a constant, arbitrarily large. On putting $\int_0^\theta \{f(t) - j/2\} dt = F(\theta)$, and using integration by parts, we get from (5.1)

$$\begin{aligned} L_1 &= \frac{F(t)}{t} \frac{\sin(n+1)(t/2) - 2^{-1} \sin(n+1)t}{t} \Bigg|_0^{c/n} \\ &\quad - (n+1) \int_0^{c/n} F(t) \frac{\cos((n+1)t/2) - \cos(n+1)t}{2t^2} dt \\ &\quad + 2 \int_0^{c/n} F(t) \frac{\sin((n+1)t/2) - 2^{-1} \sin(n+1)t}{t^3} dt \\ &= o(n) + o(n) \cdot \int_0^{c/n} t^{-1} |\sin(3(n+1)t/4) \sin((n+1)t/4)| dt \\ &\quad + o(1) \int_0^{c/n} t^{-2} \sin^2((n+1)t/4) dt = o(n), \end{aligned}$$

as $n \rightarrow \infty$. Furthermore $|L_2| < \int_{c/n}^{\pi} |f(t) - j/2| 2 \sin^2 ((n+1)t/4)/t^2 dt$; we use here Fejér's inequality, $\sin^2 x < (2x/(1+x))^2$ for $x > 0$. Then

$$\begin{aligned} |L_2| &< \frac{(n+1)^2}{2} \int_{c/n}^{\pi} |f(t) - j/2| \frac{dt}{(1 + ((n+1)t/4))^2} \\ &= \frac{(n+1)^2}{2} \left[\frac{\phi(t)}{(1 + ((n+1)t/4))^2} \right]_{c/n}^{\pi} \\ &\quad + \frac{(n+1)^2}{2} \int_{c/n}^{\pi} \frac{(n+1)\phi(t)dt}{4(1 + ((n+1)t/4))^3}, \end{aligned}$$

where $\phi(\theta) = \int_0^{\theta} |f(t) - j/2| dt$. But from (5.1), $\phi(\theta) < \gamma\theta$, γ an absolute constant, hence

$$\begin{aligned} |L_2| &< O(1) + \frac{(n+1)^3}{8} \gamma \int_{c/n}^{\pi} \frac{tdt}{(1 + ((n+1)t/4))^3} \\ &< O(1) + \frac{(n+1)^2}{2} \gamma \int_{c/n}^{\pi} \frac{dt}{(1 + ((n+1)t/4))^2} < \frac{2\gamma(n+1)}{1 + c/4} + O(1), \end{aligned}$$

and $\limsup_{n \rightarrow \infty} n^{-1} L_2 \leq 8\gamma/(4+c)$. Thus $\limsup_{n \rightarrow \infty} n^{-1} |I_n| \leq 8\gamma/(4+c)$; but c being arbitrary, now (5.8) follows.

Similarly we decompose the left side of (5.9) into $\int_0^{c/n} + \int_{c/n}^{\pi} = M_1 + M_2$ say, where

$$\begin{aligned} M_1 &= \int_0^{c/n} \{f(t) - j/2\} t^{-3} \left[\frac{n}{2} \left(\cos n \frac{t}{2} - \cos (n+2) \frac{t}{2} \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\cos (2n+1) \frac{t}{2} - \cos \frac{t}{2} \right) \right] dt \\ &\equiv \int_0^{c/n} \{f(t) - j/2\} t^{-3} Q_n(t) dt \\ &= F(t) t^{-3} Q_n(t) \Big|_0^{c/n} - \int_0^{c/n} F(t) t^{-3} Q_n'(t) dt \\ &\quad + 3 \int_0^{c/n} t^{-4} Q_n(t) F(t) dt. \end{aligned}$$

The mean value theorem yields

$$\frac{Q_n(t)}{n^4 t^4} \sim \frac{Q_n'(t)}{4n^4 t^3} \sim \frac{Q_n''(t)}{12n^4 t^2} \sim \frac{Q_n'''(t)}{24n^4 t} \sim \frac{1}{96}$$

for $t \rightarrow 0$, hence, using (5.1), we find that $M_1 = o(n^2)$, as $n \rightarrow \infty$.

To estimate M_2 we use the formula

$$n \sin x - \sin nx = \sin x \sum_{\nu=1}^n \{1 - \cos(n - 2\nu + 1)x\} > 0, \quad 0 < x < \pi,$$

which follows from

$$\frac{\sin nx}{\sin x} = \frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}} = \sum_1^n e^{i(n-2\nu+1)x} = \sum_1^n \cos(n - 2\nu + 1)x.$$

Now

$$\begin{aligned} |M_2| &< \int_{c/n}^{\pi} |f(t) - j/2| t^{-3} \sin \frac{t}{2} \sum_1^n \left\{1 - \cos(n - 2\nu + 1) \frac{t}{2}\right\} dt \\ &< \int_{c/n}^{\pi} |f(t) - j/2| t^{-2} \sum_{2\nu \leq n} \left\{1 - \cos(n - 2\nu + 1) \frac{t}{2}\right\} dt = \sum_{2\nu \leq n} D_\nu \end{aligned}$$

say, and, as in the estimate of L_2

$$\begin{aligned} D_\nu &< \phi(\pi) \frac{(n - 2\nu + 1)^2}{2(1 + (n - 2\nu + 1)\pi/4)^2} + \frac{2\gamma(n - 2\nu + 1)}{1 + c/4} \\ &< \phi(\pi) + \frac{8\gamma(n - 2\nu + 1)}{4 + c}. \end{aligned}$$

Hence $|M_2| < n\phi(\pi) + 4\gamma n^2/(4+c)$, and $\limsup_{n \rightarrow \infty} |M_2|/n^2 \leq 4\gamma/(4+c)$; c being arbitrarily large, we finally get (5.9), and Theorem 8 is proved.

6. The jump of $f(\theta)$ and Gibbs' phenomenon. The foregoing results enable us to give new formulae for the jump of $f(\theta)$ and to prove a Gibbs phenomenon.

THEOREM 9. *Under the assumptions (5.1), j is determined by any one of the formulae*

$$(6.1) \quad (C, 2) \lim nb_n = \frac{1}{\pi} j,$$

$$(6.2) \quad \sum_1^n b_\nu \sin \nu \frac{\pi}{n} \rightarrow \frac{1}{\pi} j \int_0^\pi \frac{\sin t}{t} dt,$$

$$(6.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_1^n s_\nu \left(\frac{\pi}{n} \right) = \frac{1}{\pi} j \int_0^\pi \frac{\sin t}{t} dt,$$

$$(6.4) \quad \lim_{n \rightarrow \infty} \frac{n}{2\pi} \int_0^{2\pi/n} s_n(t) dt = \frac{1}{\pi} j \int_0^{2\pi} \frac{\sin t}{t} dt,$$

where $s_n(t) = \sum_1^n b_\nu \sin \nu t$.

(6.1) is the statement of Theorem 8; (6.2) follows from Theorem 2 for $\theta_n \equiv 0$, $\phi_n = \pi/n$, $\tau_n = nb_n$, using (6.1). Similarly (6.3) follows from Theorem 4 for $\tau_n = nb_n$, $\theta_n = \pi/n$, and (6.4) follows from Theorem 7 for $\theta_n = 2\pi/n$.

To prove a Gibbs' phenomenon under the same assumptions, put, in Theorem 2, $\tau_n = nb_n$, $\theta_n \equiv 0$, $n\phi_n = \pi + O(1/n)$, then

$$s_n(\phi_n) \rightarrow \frac{1}{\pi} j \int_0^\pi t^{-1} \sin t dt,$$

hence

$$\limsup s_n(\theta_n) \geq j/2 \times 1.08949 \dots, \quad \theta_n \downarrow 0.$$

Similarly from Theorem 4 for $\alpha = \pi$

$$\frac{1}{n} \sum_1^n s_\nu(\theta_\nu) \rightarrow \frac{1}{\pi} j \int_0^\pi t^{-1} \sin t dt,$$

hence

$$\limsup s_n(\theta_n) \geq \frac{1}{\pi} j \int_0^\pi t^{-1} \sin t dt.$$

Theorem 7 also proves the presence of a Gibbs' phenomenon, however with a smaller constant.

We may also combine Theorem 8 with Theorem 3 of [3], putting there $\rho_n \equiv 1$, $\kappa = 2$, $n\theta_n = \pi + O(n^{-1})$. Summarizing, we have the following theorem.

THEOREM 10. *Under the assumptions (5.1)*

$$\limsup_{\theta_n \downarrow 0} \sum_1^n b_\nu \sin \nu \theta_n \geq \frac{jg}{2},$$

where

$$g = \frac{2}{\pi} \int_0^\pi t^{-1} \sin t dt = 1.08949 \dots$$

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